Subgraphs of large connectivity and chromatic number

Resolving a problem raised by Norin, we show that for each $k \in \mathbb{N}$, there exists an $f(k) \leq 7k$ such that every graph G with chromatic number at least $f(k)+1$ contains a subgraph H with both connectivity and chromatic number at least k. This result is best-possible up to multiplicative constants, and sharpens earlier results of Alon-Kleitman-Thomassen-Saks-Seymour from 1987 showing that $f(k) = O(k^3)$, and of Chudnovsky-Penev-Scott-Trotignon from 2013 showing that $f(k) = O(k^2)$. Our methods are robust enough to handle list colouring as well: we also show that for each $k \in \mathbb{N}$, there exists an $f_\ell(k) \leq 4k$ such that every graph G with list chromatic number at least $f_\ell(k)+1$ contains a subgraph H with both connectivity and list chromatic number at least k. This result is again best-possible up to multiplicative constants; here, unlike with $f(\cdot)$, even the existence of $f_\ell(\cdot)$ appears to have been previously unknown.